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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH TWO-POINT BOUNDARY CONDITIONS

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Abstract. The sufficient conditions are established for the existence of solutions for a class of nonlocal boundary value problems for fractional differential equations involving the Caputo fractional derivative.

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1. Introduction

Differential equations of fractional order have proved to be valuable tools in the modeling of many phenomena is various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity (Bagley, 1983; Catania & Sorrentino, 2006; Sorrentinos, 2007), dynamical processes in self-similar structures (Magin, 2004), biosciences (Lakshmikantham & Vatsala, 2008), signal processing (Oldham, 2010), system control theory (Tu *et al.*,1993; Vinagre *et al.*, 2000), electrochemistry (Metzler & Klafter, 2000) and diffusion processes (Mainardi, 1997). Further, fractional calculus has found many applications in classical mechanics (Ortigueira, 2003; Dhaigude & Rizqan, 2017) and the calculus of variations (Agrawal, 2002) and is a very useful and means for obtaining solutions to non-homogenous linear ordinary and partial differential equations. For more details we refer the reader to (Sorrentinos, 2007).

There are several approaches to fractional derivatives such as Riemann-Lowville, Caputo, Weyl, Hadamar and Grunwald-Letnikov, etc. Applied problems require those definitions of a fractional derivative that allow the utilization of physically interpretable initial and boundary conditions. The Caputo fractional derivative satisfies these demands, while the Riemann-Lowville derivative is not suitable for mixed boundary conditions.

Recently, the theory on existence and uniqueness of solutions of linear and nonlinear fractional differential equations has attracted the attention of the many authors, see for example, (Agarwal *et al.*, 2010; Ashyralyev & Sharifov, 2012; Sharifov, 2012, Ibrahim & Momani, 2007, Khan 2005, Khan *et al.*, 2011, Benchohra *et al.*, 2008; Rabei & Alhalhol, 2004) and references therein. However, many of the physical systems can better be described by nonlocal boundary conditions. Nonlocal boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering and cellular systems. Moreover, nonlocal boundary value problems a very interesting and important class of problems. They include two-point, three-point, multi-point and nonlocal boundary value problems as special cases, see (Ahmad & Nieto, 2009; Boucherif, 2009; Rabei & Alhalhol, 2004, Xinwei & Landong, 2007).

In this paper, we study existence and uniqueness of nonlinear fractional differential equations of the type

$$ {}^{c}D_{0+}^{\alpha}x(t)=f\left(t,x(t)\right),\text{for } t\in[0,T], \tag{1}$$

subject to two-point boundary conditions

$$ Ax(0)+Bx(T)=C, \tag{2}$$

where $0<\alpha<1$, ${}^{c}D_{0+}^{\alpha}$ is the Caputo fractional derivatives.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. By $C([0,T],R^n)$ we denote the Banach space of all continuous functions from $[0,T]$ into R^n with the norm $\|x\|=\max\{|x(t)|:t\in[0,T]\}$.

Definition 1. If $g\in C([a,b])$ and $\alpha>0$, then the Riemann-Lowville fractional integral is defined by

$$ I_{a+}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{g(s)}{(t-s)^{1-\alpha}}ds, \tag{3}$$

where $\Gamma(\cdot)$ is the Gamma function defined for any complex number z as

$$\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}dt.$$

Definition 2.The Caputo fractional derivative of order $\alpha>0$ of a continuous function $g:(a,b)\to R$ is defined by

$$ {}^{c}D_{a+}^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds, \tag{4}$$

where $n=[\alpha]+1$, (the notation $[\alpha]$ stands for the largest integer not greater than α).

Remark 1. Under natural conditions on $g(t)$, the Caputo fractional derivative becomes the conventional integer order derivative of the function $g(t)$ as $\alpha\to n$.

Remark 2. Let $\alpha,\beta>0$ and $n=[\alpha]+1$, then the following relations hold:

$ {}^{c}D_{0+}^{\alpha}t^{\beta}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}t^{\beta-\alpha}, \beta>n$, and ${}^{c}D_{0+}^{\alpha}t^{k}=0$, $k=0,1,2,\cdots,n-1$.

Lemma 1. For $\alpha>0$, $g(t)\in C(0,1)\cap L(0,1)$, the homogenous fractional differential equation

$$ {}^{c}D_{0+}^{\alpha}g(t)=0,$$

has a solution

$$g(t)=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1},$$

where, $c_i\in R$, $i=0,1,\cdots,n-1$, and $n=[\alpha]+1$.

Lemma 2. Assume that $g(t)\in C(0,1)\cap L(0,1)$, with derivative of order n that belongs to $C(0,1)\cap L(0,1)$, then

$$I_{0+}^{\alpha}\,{}^{c}D_{0+}^{\alpha}g(t)=g(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1},$$

where, $c_i\in R$, $i=0,1,...,n-1$, and $n=[\alpha]+1$.

Lemma 3. (Kilbas *et al*., (2006)) Let $p,q\geq 0$, $f\in L_1[0,T]$. Then

$$I_{0+}^{p}I_{0+}^{q}f(t)=I_{0+}^{p+q}f(t)=I_{0+}^{q}I_{0+}^{p}f(t) \tag{5}$$

is satisfied almost everywhere on $[0,T]$. Moreover, if $f\in C[0,T]$, then (5) is true for all $t\in[0,T]$.

Lemma 4. (Kilbas *et al*., (2006) If $q>0$, $f\in C([0,T])$, then ${}^{c}D_{0+}^{\alpha}I_{0+}^{\alpha}f(t)=f(t)$ for all $t\in[0,T]$.

3. Main Results

Lemma 5. Let $0<\alpha\leq 1$ and $f,g\in C([0,T],R^n)$. Then the unique solution of the boundary value problem for fractional differential equation

$${}^{c}D_{0+}^{\alpha}x(t)=y(t),\ t\in[0,T] \tag{6}$$

$$Ax(0)+Bx(T)=C \tag{7}$$

is given by

$$x(t)=\int_0^T G(t,s)y(s)ds+V, \tag{8}$$

where,

$$G(t,s)=\begin{cases}\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}-\frac{1}{\Gamma(\alpha)}(A+B)^{-1}B(T-s)^{\alpha-1}, 0\leq s\leq t,\\ -\frac{1}{\Gamma(\alpha)}(A+B)^{-1}B(T-s)^{\alpha-1}, t\leq s\leq T,\end{cases} \tag{9}$$

$V=(A+B)^{-1}C$.

Proof. Assume that x is a solution of the boundary value problem (6), (7), then using Lemma 2, we have

$$x(t)=I_{0+}^{\alpha}y(t)+c_1, c_1\in R^n. \tag{10}$$

From (6) and (9), we obtain

$$Ac_1+B\left(I_{0+}^{\alpha}y(T)+c_1\right)=C, \tag{11}$$

which implies that

$$c_1=(A+B)^{-1}C-(A+B)^{-1}BI_{0+}^{\alpha}y(T). \tag{12}$$

Using (11) in (10), we obtain

$$x(t)=I_{0+}^{\alpha}y(t)+(A+B)^{-1}C-(A+B)^{-1}BI_{0+}^{\alpha}y(T),$$

which can be written as (8). Lemma is provided.

Lemma 6. Assume that $f\in C([0,T]\times R^n,R^n)$, then the function $x(t)$ is solution of fractional boundary value problem (1), (2) if and only if $x(t)$ is solution of the fractional integral equation

$$x(t)=\int_0^T G(t,s)f(s,x(s))ds+(A+B)^{-1}C\,. \tag{13}$$

Proof. Let $x(t)$ be a solution of the boundary value problem (1), (2), then by same method as used in Lemma 5, we can prove that it is a solution of the fractional integral equation (13).

Conversely, let $x(t)$ satisfy (13) and denote the right hand side of equation (13) by $v(t)$. Then, by Lemmas 3 and 4, we obtain

$$v(t)=\int_0^T G(t,s)f(s,x(s))ds+(A+B)^{-1}C=$$
$$=I_{0+}^{\alpha}f(t,x(t))+(A+B)^{-1}C,$$

this implies that

$${}^{c}D_{0+}^{\alpha}v(t)={}^{c}D_{0+}^{\alpha}I_{0+}^{\alpha}f(t,x(t))+{}^{c}D_{0+}^{\alpha}C=f(t,x(t)).$$

Hence, $x(t)$ is a solution of fractional differential equation (1).Also, it is easy to verify that satisfy the condition (2). Our first result is based on Banach fixed point theorem.

Theorem 1. Assume that:

(H1) There exists a constant $L>0$ such that

$$|f(t,x)-f(t,y)|\le L|x-y|,\ \text{for each } t\in[0,T] \text{ and all } x,y\in R^n\,.$$

If

$$\frac{1}{\Gamma(\alpha+1)}\left[LT^{\alpha}\left(1+\left\|(A+B)^{-1}B\right\|\right)\right]<1 \tag{14}$$

then the boundary value problem (1)-(2) has unique solution on $[0,T]$.

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator

$$P:C([0,T],R^n)\to C([0,T],R^n)$$

defined by

$$P(x)(t)=\int_0^T G(t,s)f(s,x(s))ds+(A+B)^{-1}C. \tag{15}$$

Clearly, the fixed points of the operator P are solution of the problem (1)-(2). We shall use the Banach contraction principle to prove that P defined by (14) has a fixed point. We shall show that P is a contraction.

Let $x,y\in C([0,T],R^n)$. Then, for each $t\in[0,T]$ we have

$$|P(x)(t)-P(y)(t)|\le\int_0^T|G(t,s)||f(s,x(s))-f(s,y(s))|ds\le$$
$$\le\frac{1}{\Gamma(\alpha)}\left[\int_0^t(t-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds+\right.$$
$$\left.+\left\|(A+B)^{-1}B\right\|\int_0^T(T-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds\right]\le$$
$$\le\left\{\frac{1}{\Gamma(\alpha+1)}\left[LT^{\alpha}\left(1+\left\|(A+B)^{-1}B\right\|\right)\right]\right\}\|x-y\|\,.$$

Thus

$$\|P(x)(t)-P(y)(t)\| \le \left\{\frac{1}{\Gamma(\alpha+1)}\left[LT^{\alpha}\left(1+\left\|(A+B)^{-1}B\right\|\right)\right]\right\}\|x-y\|.$$

Consequently by (14) P is a contraction. As a consequence of Banach fixed point theorem, we deduce that P has a fixed point which is a solution of the problem (1)-(2). Theorem is provided.

The second result is based on Schaefer's fixed point theorem.

Theorem 2. Assume that:

(H2) The function $f:[0,T]\times R^n \to R^n$ is continuous.

(H3) There exists a constant $N_1 > 0$ such that

$|f(t,x)| \le N_1$ for each $t\in[0,T]$ and all $x \in R^n$.

Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.

Proof. We shall use Schaefer's fixed point theorem to prove that P defined by (15) has a fixed point. The proof will be given in several steps.

Step 1: Operator P is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C([0,T],R^n)$. Then for each $t\in[0,T]$

$$|P(x_n)(t)-P(x)(t)| \le \int_0^T |G(t,s)|\,|f(s,x_n(s))-f(s,x(s))|ds \le$$

$$\le \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}\max|f(s,x_n(s))-f(s,x(s))|ds+\right.$$

$$\left.+\left\|(A+B)^{-1}B\right\|\int_0^T (T-s)^{\alpha-1}\max|f(s,x_n(s))-f(s,x(s))|ds\right] \le$$

$$\le \frac{1}{\Gamma(\alpha+1)}\left[LT^{\alpha}\left(1+\left\|(A+B)^{-1}B\right\|\right)\right]\|f(s,x_n(s))-f(s,x_n(s))\|.$$

Since f is continuous function, we have

$$\|P(x_n)(t)-P(x)(t)\| \le \frac{1}{\Gamma(\alpha+1)}\left[LT^{\alpha}\left(1+\left\|(A+B)^{-1}B\right\|\right)\right]\|f(s,x_n(s))-f(s,x_n(s))\| \to 0$$

as $n\to\infty$.

Step 2: P maps bounded sets in bounded sets in $C([0,T],R^n)$. Indeed, it is enough to show that for any $\eta>0$, there exists a positive constant l such that for each $x\in B_\eta=\{x\in C([0,T],R^n):\|x\|\le\eta\}$, we have $\|P(x)\|\le l$. By (H4) and (H4) we have for each $t\in[0,T]$,

$$|P(x)(t)| \le \int_0^T |G(t,s)|\,|f(s,x(s))|ds+\left\|(A+B)^{-1}C\right\|.$$

Hence,

$$|P(x)(t)| \le \frac{N_1T\alpha}{\Gamma(\alpha+1)}\left[1+\left\|(A+B)^{-1}B\right\|\right]+\left\|(A+B)^{-1}C\right\|.$$

Thus

$$\|P(x)(t)\| \le \frac{N_1T\alpha}{\Gamma(\alpha+1)}\left[1+\left\|(A+B)^{-1}B\right\|\right]+\left\|(A+B)^{-1}C\right\|=l.$$

Step 3: P maps bounded sets into equicontinuous sets of $C([0,T],R^n)$. Let $t_1,t_2 \in (0,T]$, $t_1 < t_2$, B_η be a bounded set of $C([0,T],R^n)$ as in Step 2, and let $x \in B_\eta$. Then

$$|P(x)(t_2) - P(x)(t_1)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] f(s, x(s)) ds + \right.$$

$$\left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s)) ds \le \frac{N_1}{\Gamma(\alpha+1)} \left[2(t_2 - t_1)^\alpha + (t_2^\alpha - t_1^\alpha) \right] \right..$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that the operator $P: C([0,T],R^n) \to C([0,T],R^n)$ is completely continuous.

Step 4: A priori bounds. Now it remains to show that the set

$$\Delta = \{ x \in C([0,T],R^n) : x = \lambda P(x), \text{ for some } 0 < \lambda < 1 \}$$

is bounded.

Let $x \in \Delta$ then $x = \lambda(Px)$ for some $0 < \lambda < 1$ Thus, for each $t \in [0,T]$ we have

$$x(t) = \lambda \left[\int_0^T G(t,s) f(s, x(s)) ds + (A+B)^{-1} C \right].$$

This implies by (H4) and (H6) (as in step2) that for each $t \in [0,T]$ we have

$$|P(x)(t)| \le \frac{N_1 T \alpha}{\Gamma(\alpha+1)} \left[1 + \left\| (A+B)^{-1} B \right\| \right] + \left\| (A+B)^{-1} C \right\|.$$

Thus for every $t \in [0,T]$, we have

$$\|x\| \le \frac{N_1 T \alpha}{\Gamma(\alpha+1)} \left[1 + \left\| (A+B)^{-1} B \right\| \right] + \left\| (A+B)^{-1} C \right\| = R.$$

This shows that the set Δ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that P has a fixed point which is a solution of the problem (1)-(2).

In the following theorem we shall give an existence result for the problem (1)-(2) by means of an application of a Leray-Schauder type nonlinear alternative, where the condition (H3) and is weakened.

Theorem 3. Assume that (H3) and the following conditions hold.

(H4) There exist $\theta_f \in L_1([0,T],R^+)$ and continuous and no decreasing $\psi_f : [0,\infty) \to [0,\infty)$ such that

$|f(t,x)| \le \theta_f(t) \psi_f(|x|)$ for each $t \in [0,T]$ and all $x \in R.$

(H5) There exists a number $K > 0$ such that

$$\frac{K}{\psi(K) \left[\left\| I^\alpha \theta_f \right\|_{L_1} + (I^\alpha \theta_f)(T) \right] + \psi_g(K) \left\| (A+B)^{-1} \right\| \|\theta\|_{L_1}} > 1.$$

Then boundary value problem (1.1)-(1.2) has at least one solution on $[0,T]$.

Proof. Consider the operator P defined in Theorems 1 and 2. It can be easily shown that P is continuous and completely continuous. For $\lambda \in [0,1]$ let x be such that for each $t \in [0,T]$ we have $x(t) = \lambda(Px)(t)$. Then from (H4) we have for each $t \in [0,T]$

$$|x(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\theta_f(s)\psi(|x(s)|)ds +$$

$$+\frac{1}{\Gamma(\alpha)}\left\|(A+B)^{-1}B\right\|\int_0^T (T-s)^{\alpha-1}\theta_f(s)\psi_f(|x(s)|)ds \le$$

$$\le \psi_f(\|x\|)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\theta_f(s)ds +$$

$$+\psi_f(\|x\|)\frac{1}{\Gamma(\alpha)}\left\|(A+B)^{-1}B\right\|\int_0^T (T-s)^{\alpha-1}\theta_f(s)ds.$$

Thus

$$\frac{\|x\|}{\psi_f(\|x\|)\left[\left\|I^\alpha\theta_f\right\|_{L_1}+\left(I^\alpha\theta_f\right)(T)\right]} \le 1.$$

Then, by condition (H5), there exists K such that $\|x\| \ne K$.

Let

$$U = \{x \in C([0,T],R) : \|x\| < K\}.$$

The operator $P : \overline{U} \to C([0,T],R)$ is continuous and completely continuous. By the choice of U, there exists no $x \in \partial U$ such that $x = \lambda P(x)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type (Granas & Dugundji, 2013), we deduce that P has a fixed point x in $\overline{U}$, which is a solution of the problem (1)-(2). This completes of proof.

4. An example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following nonlocal boundary value problem for system fractional differential equation

$$\begin{cases} {}^cD^\alpha x_1(t) = \dfrac{1}{10}\sin x_2, t \in [0,1], 0 < \alpha < 1, \\ {}^cD^\alpha x_2(t) = \dfrac{|x_1|}{(9+e^t)(1+|x_1|)} \end{cases}, \tag{16}$$

$$x_1(0) = 0,\ x_2(1) = 1. \tag{17}$$

Evidently, $A + B = E$, $\left\|(A+B)^{-1}\right\| = 1$ and $\left\|(A+B)^{-1}B\right\| = 1$.

Hence the conditions (H1)-(H2) hold with $L = 0{,}1$. We shall check that condition (14) is satisfied for appropriate values of $0 < \alpha \le 1$ with $T = 1$. Indeed

$$\frac{0{,}2}{\Gamma(\alpha+1)} < 1. \tag{18}$$

Then by Theorem 3.3 the boundary value problem (16)-(17) has a unique solution on $[0,1]$ for values of α satisfying condition (18). For example, if $\alpha = 0{,}2$ then $\Gamma(\alpha+1) = \Gamma(1{,}2) = 0{,}92$ and $\dfrac{0{,}2}{\Gamma(\alpha+1)} = 0{,}10869 < 1$.

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